

STABLY CAYLEY SEMISIMPLE GROUPS

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ABSTRACT. A linear algebraic group G over a field k is called a Cayley group if it admits a Cayley map, i.e., a G -equivariant birational isomorphism over k between the group variety G and its Lie algebra $\mathrm{Lie}(G)$. A prototypical example is the classical “Cayley transform” for the special orthogonal group \mathbf{SO}_n defined by Arthur Cayley in 1846. A linear algebraic group G is called stably Cayley if $G \times S$ is Cayley for some split k -torus S . We classify stably Cayley semisimple groups over an arbitrary field k of characteristic 0.

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To Alexander Merkurjev on the occasion of his 60th birthday

0 INTRODUCTION

Let k be a field of characteristic 0 and \bar{k} a fixed algebraic closure of k . Let G be a connected linear algebraic k -group. A birational isomorphism $\phi: G \xrightarrow{\sim} \mathrm{Lie}(G)$ is called a *Cayley map* if it is equivariant with respect to the conjugation action of G on itself and the adjoint action of G on its Lie algebra $\mathrm{Lie}(G)$, respectively. A linear algebraic k -group G is called *Cayley* if it admits a Cayley map, and *stably Cayley* if $G \times_k (\mathbb{G}_{m,k})^r$ is Cayley for some $r \geq 0$. Here $\mathbb{G}_{m,k}$ denotes the multiplicative group over k . These notions were introduced by Lemire, Popov and Reichstein [LPR]; for a more detailed discussion and numerous classical examples we refer the reader to [LPR, Introduction]. The main results of [LPR] are the classifications of Cayley and stably Cayley simple groups over an algebraically closed field k of characteristic 0. Over an arbitrary field k of characteristic 0 stably Cayley simple k -groups, stably Cayley simply connected semisimple k -groups and stably Cayley adjoint semisimple k -groups were classified in the paper [BKLR] of Borovoi, Kunyavskiĭ, Lemire and Reichstein. In

the present paper, building on results of [LPR] and [BKLR], we classify all stably Cayley *semisimple* k -groups (not necessarily simple, or simply connected, or adjoint) over an arbitrary field k of characteristic 0.

By a semisimple (or reductive) k -group we always mean a *connected* semisimple (or reductive) k -group. We shall need the following result of [BKLR] extending [LPR, Theorem 1.28].

THEOREM 0.1 ([BKLR, Theorem 1.4]). *Let k be a field of characteristic 0 and G an absolutely simple k -group. Then the following conditions are equivalent:*

- (a) G is stably Cayley over k ;
- (b) G is an arbitrary k -form of one of the following groups:

$$\mathbf{SL}_3, \mathbf{PGL}_2, \mathbf{PGL}_{2n+1} \ (n \geq 1), \mathbf{SO}_n \ (n \geq 5), \mathbf{Sp}_{2n} \ (n \geq 1), \mathbf{G}_2,$$

or an inner k -form of $\mathbf{PGL}_{2n} \ (n \geq 2)$.

In this paper we classify stably Cayley semisimple groups over an *algebraically closed* field k of characteristic 0 (Theorem 0.2) and, more generally, over an *arbitrary* field k of characteristic 0 (Theorem 0.3). Note that Theorem 0.2 was conjectured in [BKLR, Remark 9.3].

THEOREM 0.2. *Let k be an algebraically closed field of characteristic 0 and G a semisimple k -group. Then G is stably Cayley if and only if G decomposes into a direct product $G_1 \times_k \cdots \times_k G_s$ of its normal subgroups, where each G_i ($i = 1, \dots, s$) either is a stably Cayley simple k -group (i.e., isomorphic to one of the groups listed in Theorem 0.1) or is isomorphic to the stably Cayley semisimple k -group \mathbf{SO}_4 .*

THEOREM 0.3. *Let G be a semisimple k -group over a field k of characteristic 0 (not necessarily algebraically closed). Then G is stably Cayley over k if and only if G decomposes into a direct product $G_1 \times_k \cdots \times_k G_s$ of its normal k -subgroups, where each G_i ($i = 1, \dots, s$) is isomorphic to the Weil restriction $R_{l_i/k} G_{i,l_i}$ for some finite field extension l_i/k , and each G_{i,l_i} is either a stably Cayley absolutely simple group over l_i (i.e., one of the groups listed in Theorem 0.1) or an l_i -form of the semisimple group \mathbf{SO}_4 (which is always stably Cayley, but is not absolutely simple and can be not l_i -simple).*

Note that the “if” assertions in Theorems 0.2 and 0.3 follow immediately from the definitions.

The rest of the paper is structured as follows. In Section 1 we recall the definition of a quasi-permutation lattice and state some known results, in particular, an assertion from [LPR, Theorem 1.27] that reduces Theorem 0.2 to an assertion on lattices. In Sections 2 and 3 we construct certain families of non-quasi-permutation lattices. In particular, we correct an inaccuracy in [BKLR]; see Remark 2.5. In Section 4 we prove (in the language of lattices) Theorem

0.2 in the special case when G is isogenous to a direct product of simple groups of type \mathbf{A}_{n-1} with $n \geq 3$. In Section 5 we prove (again in the language of lattices) Theorem 0.2 in the general case. In Section 6 we deduce Theorem 0.3 from Theorem 0.2. In Appendix A we prove in terms of lattices only, that certain quasi-permutation lattices are indeed quasi-permutation.

1 PRELIMINARIES ON QUASI-PERMUTATION GROUPS AND ON CHARACTER LATTICES

In this section we gather definitions and known results concerning quasi-permutation lattices, quasi-invertible lattices and character lattices that we need for the proofs of Theorems 0.2 and 0.3. For details see [BKLR, Sections 2 and 10] and [LPR, Introduction].

1.1. By a *lattice* we mean a pair (Γ, L) where Γ is a finite group acting on a finitely generated free abelian group L . We say also that L is a Γ -lattice. A Γ -lattice L is called a *permutation* lattice if it has a \mathbb{Z} -basis permuted by Γ . Following Colliot-Thélène and Sansuc [CTS], we say that two Γ -lattices L and L' are *equivalent*, and write $L \sim L'$, if there exist short exact sequences

$$0 \rightarrow L \rightarrow E \rightarrow P \rightarrow 0 \quad \text{and} \quad 0 \rightarrow L' \rightarrow E \rightarrow P' \rightarrow 0$$

with the same Γ -lattice E , where P and P' are permutation Γ -lattices. For a proof that this is indeed an equivalence relation see [CTS, Lemma 8, p. 182] or [Sw, Section 8]. Note that if there exists a short exact sequence of Γ -lattices

$$0 \rightarrow L \rightarrow L' \rightarrow Q \rightarrow 0$$

where Q is a permutation Γ -lattice, then, taking in account the trivial short exact sequence

$$0 \rightarrow L' \rightarrow L' \rightarrow 0 \rightarrow 0,$$

we obtain that $L \sim L'$. If Γ -lattices L, L', M, M' satisfy $L \sim L'$ and $M \sim M'$, then clearly $L \oplus M \sim L' \oplus M'$.

Definition 1.2. A Γ -lattice L is called a *quasi-permutation* lattice if there exists a short exact sequence

$$0 \rightarrow L \rightarrow P \rightarrow P' \rightarrow 0, \tag{1.1}$$

where both P and P' are permutation Γ -lattices.

LEMMA 1.3 (well-known). *A Γ -lattice L is quasi-permutation if and only if $L \sim 0$.*

Proof. If L is quasi-permutation, then sequence (1.1) together with the trivial short exact sequence

$$0 \rightarrow 0 \rightarrow P \rightarrow P \rightarrow 0$$

shows that $L \sim 0$. Conversely, if $L \sim 0$, then there are short exact sequences

$$0 \rightarrow L \rightarrow E \rightarrow P \rightarrow 0 \quad \text{and} \quad 0 \rightarrow 0 \rightarrow E \rightarrow P' \rightarrow 0,$$

where P and P' are permutation lattices. From the second exact sequence we have $E \cong P'$, hence E is a permutation lattice, and then the first exact sequence shows that L is a quasi-permutation lattice. \square

Definition 1.4. A Γ -lattice L is called *quasi-invertible* if it is a direct summand of a quasi-permutation Γ -lattice.

Note that if a Γ -lattice L is not quasi-invertible, then it is not quasi-permutation.

LEMMA 1.5 (well-known). *If a Γ -lattice L is quasi-permutation (resp., quasi-invertible) and $L' \sim L$, then L' is quasi-permutation (resp., quasi-invertible) as well.*

Proof. If L is quasi-permutation, then using Lemma 1.3 we see that $L' \sim L \sim 0$, hence L' is quasi-permutation. If L is quasi-invertible, then $L \oplus M$ is quasi-permutation for some Γ -lattice M , and by Lemma 1.3 we have $L \oplus M \sim 0$. We see that $L' \oplus M \sim L \oplus M \sim 0$, and by Lemma 1.3 we obtain that $L' \oplus M$ is quasi-permutation, hence L' is quasi-invertible. \square

Let $\mathbb{Z}[\Gamma]$ denote the group ring of a finite group Γ . We define the Γ -lattice J_Γ by the exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z}[\Gamma] \rightarrow J_\Gamma \rightarrow 0,$$

where N is the norm map, see [BKLR, before Lemma 10.4]. We refer to [BKLR, Proposition 10.6] for a proof of the following result, due to Voskresenskiĭ [Vo1, Corollary of Theorem 7]:

PROPOSITION 1.6. *Let $\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where p is a prime. Then the Γ -lattice J_Γ is not quasi-invertible.*

Note that if $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\text{rank } J_\Gamma = 3$.

We shall use the following lemma from [BKLR]:

LEMMA 1.7 ([BKLR, Lemma 2.8]). *Let W_1, \dots, W_m be finite groups. For each $i = 1, \dots, m$, let V_i be a finite-dimensional \mathbb{Q} -representation of W_i . Set $V := V_1 \oplus \dots \oplus V_m$. Suppose $L \subset V$ is a free abelian subgroup, invariant under $W := W_1 \times \dots \times W_m$. If L is a quasi-permutation W -lattice, then for each $i = 1, \dots, m$ the intersection $L_i := L \cap V_i$ is a quasi-permutation W_i -lattice.*

We shall need the notion, due to [LPR] and [BKLR], of the character lattice of a reductive k -group G over a field k . Let \bar{k} be a separable closure of k . Let $T \subset G$ be a maximal torus (defined over k). Set $\bar{T} = T \times_k \bar{k}$, $\bar{G} = G \times_k \bar{k}$. Let $X(\bar{T})$

denote the character group of $\bar{T} := T \times_k \bar{k}$. Let $W = W(\bar{G}, \bar{T}) := \mathcal{N}_G(\bar{T})/\bar{T}$ denote the Weyl group, it acts on $X(\bar{T})$. Consider the canonical Galois action on $X(\bar{T})$, it defines a homomorphism $\text{Gal}(\bar{k}/k) \rightarrow \text{Aut } X(\bar{T})$. The image $\text{im } \rho \subset \text{Aut } X(\bar{T})$ normalizes W , hence $\text{im } \rho \cdot W$ is a subgroup of $\text{Aut } X(\bar{T})$. By the character lattice of G we mean the pair $\mathcal{X}(G) := (\text{im } \rho \cdot W, X(\bar{T}))$ (up to an isomorphism it does not depend on the choice of T). In particular, if k is algebraically closed, then $\mathcal{X}(G) = (W, X(T))$.

We shall reduce Theorem 0.2 to an assertion about quasi-permutation lattices using the following result due to [LPR]:

PROPOSITION 1.8 ([LPR, Theorem 1.27], see also [BKLR, Theorem 1.3]). *A reductive group G over an algebraically closed field k of characteristic 0 is stably Cayley if and only if its character lattice $\mathcal{X}(G)$ is quasi-permutation, i.e., $X(T)$ is a quasi-permutation $W(G, T)$ -lattice.*

We shall use the following result due to Cortella and Kunyavskii [CK] and to Lemire, Popov and Reichstein [LPR].

PROPOSITION 1.9 ([CK], [LPR]). *Let D be a connected Dynkin diagram. Let $R = R(D)$ denote the corresponding root system, $W = W(D)$ denote the Weyl group, $Q = Q(D)$ denote the root lattice, and $P = P(D)$ denote the weight lattice. We say that L is an intermediate lattice between Q and P if $Q \subset L \subset P$ (note that $L = Q$ and $L = P$ are possible). Then the following list gives (up to an isomorphism) all the pairs (D, L) such that L is a quasi-permutation intermediate $W(D)$ -lattice between $Q(D)$ and $P(D)$:*

$$Q(\mathbf{A}_n), Q(\mathbf{B}_n), P(\mathbf{C}_n), \mathcal{X}(\mathbf{SO}_{2n}) \text{ (then } D = \mathbf{D}_n),$$

or D is any connected Dynkin diagram of rank 1 or 2 (i.e. $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_2$, or \mathbf{G}_2) and L is any lattice between $Q(D)$ and $P(D)$, (i.e., either $L = P(D)$ or $L = Q(D)$).

Proof. The positive result (the assertion that the lattices in the list are indeed quasi-permutation) follows from the assertion that the corresponding groups are stably Cayley (or that their generic tori are stably rational), see the references in [CK], Section 3. See Appendix A below for a proof of this positive result in terms of lattices only. The difficult part of Proposition 1.9 is the negative result (the assertion that all the other lattices are not quasi-permutation). This was proved in [CK, Theorem 0.1] in the cases when $L = Q$ or $L = P$, and in [LPR, Propositions 5.1 and 5.2] in the cases when $Q \subsetneq L \subsetneq P$ (this can happen only when $D = \mathbf{A}_n$ or $D = \mathbf{D}_n$). \square

Remark 1.10. It follows from Proposition 1.9 that, in particular, the following lattices are quasi-permutation: $Q(\mathbf{A}_1)$, $P(\mathbf{A}_1)$, $P(\mathbf{A}_2)$, $P(\mathbf{B}_2)$, $Q(\mathbf{C}_2)$, $Q(\mathbf{G}_2) = P(\mathbf{G}_2)$, $Q(\mathbf{D}_3) = Q(\mathbf{A}_3)$, $\mathcal{X}(\mathbf{SL}_4/\mu_2) = \mathcal{X}(\mathbf{SO}_6)$.

2 A FAMILY OF NON-QUASI-PERMUTATION LATTICES

In this section we construct a family of non-quasi-permutation (even non-quasi-invertible) lattices.

2.1. We consider a Dynkin diagram $D \sqcup \Delta$ (disjoint union). We assume that $D = \bigsqcup_{i \in I} D_i$ (a finite disjoint union), where each D_i is of type \mathbf{B}_{l_i} ($l_i \geq 1$) or \mathbf{D}_{l_i} ($l_i \geq 2$) (and where $\mathbf{B}_1 = \mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{C}_2$, $\mathbf{D}_2 = \mathbf{A}_1 \sqcup \mathbf{A}_1$, and $\mathbf{D}_3 = \mathbf{A}_3$ are permitted). We denote by m the cardinality of the finite index set I . We assume that $\Delta = \bigsqcup_{\iota=1}^{\mu} \Delta_{\iota}$ (disjoint union), where Δ_{ι} is of type $\mathbf{A}_{2n_{\iota}-1}$, $n_{\iota} \geq 2$ ($\mathbf{A}_3 = \mathbf{D}_3$ is permitted). We assume that $m \geq 1$ and $\mu \geq 0$ (in the case $\mu = 0$ the diagram Δ is empty).

For each $i \in I$ we realize the root system $R(D_i)$ of type \mathbf{B}_{l_i} or \mathbf{D}_{l_i} in the standard way in the space $V_i := \mathbb{R}^{l_i}$ with basis $(e_s)_{s \in S_i}$ where S_i is an index set consisting of l_i elements; cf. [Bou, Planche II] for \mathbf{B}_l ($l \geq 2$) (the relevant formulas for \mathbf{B}_1 are similar) and [Bou, Planche IV] for \mathbf{D}_l ($l \geq 3$) (again, the relevant formulas for \mathbf{D}_2 are similar). Let $M_i \subset V_i$ denote the lattice generated by the basis vectors $(e_s)_{s \in S_i}$. Let $P_i \supset M_i$ denote the weight lattice of the root system D_i . Set $S = \bigsqcup_i S_i$ (disjoint union). Consider the vector space $V = \bigoplus_i V_i$ with basis $(e_s)_{s \in S}$. Let $M_D \subset V$ denote the lattice generated by the basis vectors $(e_s)_{s \in S}$, then $M_D = \bigoplus_i M_i$. Set $P_D = \bigoplus_i P_i$.

For each $\iota = 1, \dots, \mu$ we realize the root system $R(\Delta_{\iota})$ of type $\mathbf{A}_{2n_{\iota}-1}$ in the standard way in the subspace V_{ι} of vectors with zero sum of the coordinates in the space $\mathbb{R}^{2n_{\iota}}$ with basis $\varepsilon_{\iota,1}, \dots, \varepsilon_{\iota,2n_{\iota}}$; cf. [Bou, Planche I]. Let Q_{ι} be the root lattice of $R(\Delta_{\iota})$ with basis $\varepsilon_{\iota,1} - \varepsilon_{\iota,2}, \varepsilon_{\iota,2} - \varepsilon_{\iota,3}, \dots, \varepsilon_{\iota,2n_{\iota}-1} - \varepsilon_{\iota,2n_{\iota}}$, and let $P_{\iota} \supset Q_{\iota}$ be the weight lattice of $R(\Delta_{\iota})$. Set $Q_{\Delta} = \bigoplus_{\iota} Q_{\iota}$, $P_{\Delta} = \bigoplus_{\iota} P_{\iota}$.

Set

$$W := \prod_{i \in I} W(D_i) \times \prod_{\iota=1}^{\mu} W(\Delta_{\iota}), \quad L' = M_D \oplus Q_{\Delta} = \bigoplus_{i \in I} M_i \oplus \bigoplus_{\iota=1}^{\mu} Q_{\iota},$$

then W acts on L' and on $L' \otimes_{\mathbb{Z}} \mathbb{R}$. For each i consider the vector

$$x_i = \sum_{s \in S_i} e_s \in M_i,$$

then $\frac{1}{2}x_i \in P_i$. For each ι consider the vector

$$\xi_{\iota} = \varepsilon_{\iota,1} - \varepsilon_{\iota,2} + \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \dots + \varepsilon_{\iota,2n_{\iota}-1} - \varepsilon_{\iota,2n_{\iota}} \in Q_{\iota},$$

then $\frac{1}{2}\xi_{\iota} \in P_{\iota}$; see [Bou, Planche I]. Write

$$\xi'_{\iota} = \varepsilon_{\iota,1} - \varepsilon_{\iota,2}, \quad \xi''_{\iota} = \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \dots + \varepsilon_{\iota,2n_{\iota}-1} - \varepsilon_{\iota,2n_{\iota}},$$

then $\xi_{\iota} = \xi'_{\iota} + \xi''_{\iota}$. Consider the vector

$$v = \frac{1}{2} \sum_{i \in I} x_i + \frac{1}{2} \sum_{\iota=1}^{\mu} \xi_{\iota} = \frac{1}{2} \sum_{s \in S} e_s + \frac{1}{2} \sum_{\iota=1}^{\mu} \xi_{\iota} \in P_D \oplus P_{\Delta}.$$

Set

$$L = \langle L', v \rangle, \quad (2.1)$$

then $[L : L'] = 2$ because $v \in \frac{1}{2}L' \setminus L'$. Note that the sublattice $L \subset P_D \oplus P_\Delta$ is W -invariant. Indeed, the group W acts on $(P_D \oplus P_\Delta)/(M_D \oplus Q_\Delta)$ trivially.

PROPOSITION 2.2. *We assume that $m \geq 1$, $m + \mu \geq 2$. If $\mu = 0$, we assume that not all of D_i are of types \mathbf{B}_1 or \mathbf{D}_2 . Then the W -lattice L as in (2.1) is not quasi-invertible, hence not quasi-permutation.*

Proof. We consider a group $\Gamma = \{e, \gamma_1, \gamma_2, \gamma_3\}$ of order 4, where $\gamma_1, \gamma_2, \gamma_3$ are of order 2. The idea of our proof is to construct an embedding

$$j: \Gamma \rightarrow W$$

in such a way that L , viewed as a Γ -lattice, is equivalent to its Γ -sublattice L_1 , and L_1 is isomorphic to a direct sum of a Γ -sublattice $L_0 \simeq J_\Gamma$ of rank 3 and a number of Γ -lattices of rank 1. Since by Proposition 1.6 J_Γ is not quasi-invertible, this will imply that L_1 and L are not quasi-invertible Γ -lattices, and hence L is not quasi-invertible as a W -lattice. We shall now fill in the details of this argument in four steps.

Step 1. We begin by partitioning each S_i for $i \in I$ into three (non-overlapping) subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$, subject to the requirement that

$$|S_{i,1}| \equiv |S_{i,2}| \equiv |S_{i,3}| \equiv l_i \pmod{2} \text{ if } D_i \text{ is of type } \mathbf{D}_{l_i}. \quad (2.2)$$

We then set U_1 to be the union of the $S_{i,1}$, U_2 to be the union of the $S_{i,2}$, and U_3 to be the union of the $S_{i,3}$, as i runs over I .

LEMMA 2.3. (i) *If $\mu \geq 1$, the subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of S_i can be chosen, subject to (2.2), so that $U_1 \neq \emptyset$.*

(ii) *If $\mu = 0$ (and $m \geq 2$), the subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of S_i can be chosen, subject to (2.2), so that $U_1, U_2, U_3 \neq \emptyset$.*

To prove the lemma, first consider case (i). For all i such that D_i is of type \mathbf{D}_{l_i} with *odd* l_i , we partition S_i into three non-empty subsets of odd cardinalities. For all the other i we take $S_{i,1} = S_i$, $S_{i,2} = S_{i,3} = \emptyset$. Then $U_1 \neq \emptyset$ (note that $m \geq 1$) and (2.2) is satisfied.

In case (ii), if one of the D_i is of type \mathbf{D}_{l_i} where $l_i \geq 3$ is *odd*, then we partition S_i for each such D_i into three non-empty subsets of odd cardinalities. We partition all the other S_i as follows:

$$S_{i,1} = S_{i,2} = \emptyset \text{ and } S_{i,3} = S_i. \quad (2.3)$$

Clearly $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

If there is no D_i of type \mathbf{D}_{l_i} with odd $l_i \geq 3$, but one of the D_i , say for $i = i_0$, is \mathbf{D}_l with *even* $l \geq 4$, then we partition S_{i_0} into two non-empty subsets $S_{i_0,1}$ and

$S_{i_0,2}$ of even cardinalities, and set $S_{i_0,3} = \emptyset$. We partition the sets S_i for $i \neq i_0$ as in (2.3) (note that by our assumption $m \geq 2$). Once again, $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

If there is no D_i of type \mathbf{D}_{l_i} with $l_i \geq 3$ (odd or even), but one of the D_i , say for $i = i_0$, is of type \mathbf{B}_l with $l \geq 2$, we partition S_{i_0} into two non-empty subsets $S_{i_0,1}$ and $S_{i_0,2}$, and set $S_{i_0,3} = \emptyset$. We partition the sets S_i for $i \neq i_0$ as in (2.3) (again, note that $m \geq 2$). Once again, $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

Since by our assumption not all of D_i are of type \mathbf{B}_1 or \mathbf{D}_2 , we have exhausted all the cases. This completes the proof of Lemma 2.3. \square

Step 2. We continue proving Proposition 2.2. We construct an embedding $\Gamma \hookrightarrow W$.

For $s \in S$ we denote by c_s the automorphism of L taking the basis vector e_s to $-e_s$ and fixing all the other basis vectors. For $\iota = 1, \dots, \mu$ we define $\tau_\iota^{(12)} = \text{Transp}((\iota, 1), (\iota, 2)) \in W_\iota$ (the transposition of the basis vectors $\varepsilon_{\iota,1}$ and $\varepsilon_{\iota,2}$). Set

$$\tau_\iota^{>2} = \text{Transp}((\iota, 3), (\iota, 4)) \cdot \dots \cdot \text{Transp}((\iota, 2n_\iota - 1), (\iota, 2n_\iota)) \in W_\iota.$$

Write $\Gamma = \{e, \gamma_1, \gamma_2, \gamma_3\}$ and define an embedding $j: \Gamma \hookrightarrow W$ as follows:

$$\begin{aligned} j(\gamma_1) &= \prod_{s \in S \setminus U_1} c_s \cdot \prod_{\iota=1}^{\mu} \tau_\iota^{(12)} \tau_\iota^{>2}; \\ j(\gamma_2) &= \prod_{s \in S \setminus U_2} c_s \cdot \prod_{\iota=1}^{\mu} \tau_\iota^{(12)}; \\ j(\gamma_3) &= \prod_{s \in S \setminus U_3} c_s \cdot \prod_{\iota=1}^{\mu} \tau_\iota^{>2}. \end{aligned}$$

Note that if D_i is of type \mathbf{D}_{l_i} , then by (2.2) for $\varkappa = 1, 2, 3$ the cardinality $\#(S_i \setminus S_{i,\varkappa})$ is even, hence the product of c_s over $s \in S_i \setminus S_{i,\varkappa}$ is contained in $W(D_i)$ for all such i , and therefore, $j(\gamma_\varkappa) \in W$. Since $j(\gamma_1)$, $j(\gamma_2)$ and $j(\gamma_3)$ commute, are of order 2, and $j(\gamma_1)j(\gamma_2) = j(\gamma_3)$, we see that j is a homomorphism. If $\mu \geq 1$, then, since $2n_1 \geq 4$, clearly $j(\gamma_\varkappa) \neq 1$ for $\varkappa = 1, 2, 3$, hence j is an embedding. If $\mu = 0$, then the sets $S \setminus U_1$, $S \setminus U_2$ and $S \setminus U_3$ are nonempty, and again $j(\gamma_\varkappa) \neq 1$ for $\varkappa = 1, 2, 3$, hence j is an embedding.

Step 3. We construct a Γ -sublattice L_0 of rank 3. Write a vector $\mathbf{x} \in L$ as

$$\mathbf{x} = \sum_{s \in S} b_s e_s + \sum_{\iota=1}^{\mu} \sum_{\nu=1}^{2n_\iota} \beta_{\iota,\nu} \varepsilon_{\iota,\nu},$$

where $b_s, \beta_{\iota,\nu} \in \frac{1}{2}\mathbb{Z}$. Set $n' = \sum_{\iota=1}^{\mu} (n_\iota - 1)$. Define a Γ -equivariant homomorphism

$$\phi: L \rightarrow \mathbb{Z}^{n'}, \quad \mathbf{x} \mapsto (\beta_{\iota,2\lambda-1} + \beta_{\iota,2\lambda})_{\iota=1,\dots,\mu, \lambda=2,\dots,n_\iota}$$

(we skip $\lambda = 1$). We obtain a short exact sequence of Γ -lattices

$$0 \rightarrow L_1 \rightarrow L \xrightarrow{\phi} \mathbb{Z}^{n'} \rightarrow 0,$$

where $L_1 := \ker \phi$. Since Γ acts trivially on $\mathbb{Z}^{n'}$, we have $L_1 \sim L$. Therefore, it suffices to show that L_1 is not quasi-invertible.

Recall that

$$v = \frac{1}{2} \sum_{s \in S} e_s + \frac{1}{2} \sum_{\iota=1}^{\mu} \xi_{\iota}.$$

Set $v_1 = \gamma_1 \cdot v$, $v_2 = \gamma_2 \cdot v$, $v_3 = \gamma_3 \cdot v$. Set

$$L_0 = \langle v, v_1, v_2, v_3 \rangle.$$

We have

$$v_1 = \frac{1}{2} \sum_{s \in U_1} e_s - \frac{1}{2} \sum_{s \in U_2 \cup U_3} e_s - \frac{1}{2} \sum_{\iota=1}^{\mu} \xi_{\iota},$$

whence

$$v + v_1 = \sum_{s \in U_1} e_s. \quad (2.4)$$

We have

$$v_2 = \frac{1}{2} \sum_{s \in U_2} e_s - \frac{1}{2} \sum_{s \in U_1 \cup U_3} e_s + \frac{1}{2} \sum_{\iota=1}^{\mu} (-\xi'_{\iota} + \xi''_{\iota}),$$

whence

$$v + v_2 = \sum_{s \in U_2} e_s + \sum_{\iota=1}^{\mu} \xi''_{\iota}. \quad (2.5)$$

We have

$$v_3 = \frac{1}{2} \sum_{s \in U_3} e_s - \frac{1}{2} \sum_{s \in U_1 \cup U_2} e_s + \frac{1}{2} \sum_{\iota=1}^{\mu} (\xi'_{\iota} - \xi''_{\iota}),$$

whence

$$v + v_3 = \sum_{s \in U_3} e_s + \sum_{\iota=1}^{\mu} \xi'_{\iota}. \quad (2.6)$$

Clearly, we have

$$v + v_1 + v_2 + v_3 = 0.$$

Since the set $\{v, v_1, v_2, v_3\}$ is the orbit of v under Γ , the sublattice $L_0 = \langle v, v_1, v_2, v_3 \rangle \subset L$ is Γ -invariant. If $\mu \geq 1$, then $U_1 \neq \emptyset$, and we see from (2.4), (2.5) and (2.6) that $\text{rank } L_0 \geq 3$. If $\mu = 0$, then $U_1, U_2, U_3 \neq \emptyset$, and again we see from (2.4), (2.5) and (2.6) that $\text{rank } L_0 \geq 3$. Thus $\text{rank } L_0 = 3$ and $L_0 \simeq J_{\Gamma}$, whence by Proposition 1.6 L_0 is not quasi-invertible.

Step 4. We show that L_0 is a direct summand of L_1 . Set $m' = |S|$.

First assume that $\mu \geq 1$. Choose $u_1 \in U_1 \subset S$. Set $S' = S \setminus \{u_1\}$. For each $s \in S'$ (i.e., $s \neq u_1$) consider the one-dimensional (i.e., of rank 1) lattice $X_s = \langle e_s \rangle$. We obtain $m' - 1$ Γ -invariant one-dimensional sublattices of L_1 .

Denote by Υ the set of pairs (ι, λ) such that $1 \leq \iota \leq \mu$, $1 \leq \lambda \leq n_\iota$, and if $\iota = 1$, then $\lambda \neq 1, 2$. For each $(\iota, \lambda) \in \Upsilon$ consider the one-dimensional lattice

$$\Xi_{\iota, \lambda} = \langle \varepsilon_{\iota, 2\lambda-1} - \varepsilon_{\iota, 2\lambda} \rangle.$$

We obtain $-2 + \sum_{\iota=1}^{\mu} n_\iota$ one-dimensional Γ -invariant sublattices of L_1 .

We show that

$$L_1 = L_0 \oplus \bigoplus_{s \in S'} X_s \oplus \bigoplus_{(\iota, \lambda) \in \Upsilon} \Xi_{\iota, \lambda}. \quad (2.7)$$

Set $L'_1 = \langle L_0, (X_s)_{s \neq u_1}, (\Xi_{\iota, \lambda})_{(\iota, \lambda) \in \Upsilon} \rangle$, then

$$\text{rank } L'_1 \leq 3 + (m' - 1) - 2 + \sum_{\iota=1}^{\mu} n_\iota = m' + \sum_{\iota=1}^{\mu} (2n_\iota - 1) - \sum_{\iota=1}^{\mu} (n_\iota - 1) = \text{rank } L_1. \quad (2.8)$$

Therefore, it suffices to check that $L'_1 \supset L_1$. The set

$$\{v\} \cup \{e_s \mid s \in S\} \cup \{\varepsilon_{\iota, 2\lambda-1} - \varepsilon_{\iota, 2\lambda} \mid 1 \leq \iota \leq \mu, 1 \leq \lambda \leq n_\iota\}$$

is a set of generators of L_1 . By construction $v, v_1, v_2, v_3 \in L_0 \subset L'_1$. We have $e_s \in X_s \subset L'_1$ for $s \neq u_1$. By (2.4) $\sum_{s \in U_1} e_s \in L'_1$, hence $e_{u_1} \in L'_1$. By construction

$$\varepsilon_{\iota, 2\lambda-1} - \varepsilon_{\iota, 2\lambda} \in L'_1, \quad \text{for all } (\iota, \lambda) \neq (1, 1), (1, 2).$$

From (2.6) and (2.5) we see that

$$\sum_{\iota=1}^{\mu} (\varepsilon_{\iota, 1} - \varepsilon_{\iota, 2}) \in L'_1, \quad \sum_{\iota=1}^{\mu} \xi''_{\iota} \in L'_1.$$

Thus

$$\varepsilon_{1, 1} - \varepsilon_{1, 2} \in L'_1, \quad \varepsilon_{1, 3} - \varepsilon_{1, 4} \in L'_1.$$

We conclude that $L'_1 \supset L_1$, hence $L_1 = L'_1$. From dimension count (2.8) we see that (2.7) holds.

Now assume that $\mu = 0$. Then for each $\varkappa = 1, 2, 3$ we choose an element $u_\varkappa \in U_\varkappa$ and set $U'_\varkappa = U_\varkappa \setminus \{u_\varkappa\}$. We set $S' = U'_1 \cup U'_2 \cup U'_3 = S \setminus \{u_1, u_2, u_3\}$. Again for $s \in S'$ (i.e., $s \neq u_1, u_2, u_3$) consider the one-dimensional lattice $X_s = \langle e_s \rangle$. We obtain $m' - 3$ one-dimensional Γ -invariant sublattices of $L_1 = L$. We show that

$$L_1 = L_0 \oplus \bigoplus_{s \in S'} X_s. \quad (2.9)$$

Set $L'_1 = \langle L_0, (X_s)_{s \in S'} \rangle$, then

$$\text{rank } L'_1 \leq 3 + m' - 3 = m' = \text{rank } L_1. \quad (2.10)$$

Therefore, it suffices to check that $L'_1 \supset L_1$. The set $\{v\} \cup \{e_s \mid s \in S\}$ is a set of generators of $L_1 = L$. By construction $v, v_1, v_2, v_3 \in L'_1$ and $e_s \in L'_1$ for $s \neq u_1, u_2, u_3$. We see from (2.4), (2.5), (2.6) that $e_s \in L'_1$ also for $s = u_1, u_2, u_3$. Thus $L'_1 \supset L_1$, hence $L'_1 = L_1$. From dimension count (2.10) we see that (2.9) holds.

We see that in both cases $\mu \geq 1$ and $\mu = 0$, the sublattice L_0 is a direct summand of L_1 . Since by Proposition 1.6 L_0 is not quasi-invertible as a Γ -lattice, it follows that L_1 and L are not quasi-invertible as Γ -lattices. Thus L is not quasi-invertible as a W -lattice. This completes the proof of Proposition 2.2. \square

Remark 2.4. Since $\text{III}^2(\Gamma, J_\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ (Voskresenskii, see [BKLR, Section 10] for the notation and the result), our argument shows that $\text{III}^2(\Gamma, L) \cong \mathbb{Z}/2\mathbb{Z}$.

Remark 2.5. The proof of [BKLR, Lemma 12.3] (which is a version with $\mu = 0$ of Lemma 2.3 above) contains an inaccuracy, though the lemma as stated is correct. Namely, in [BKLR] we write that if there exists i such that Δ_i is of type \mathbf{D}_{l_i} where $l_i \geq 3$ is odd, then we partition S_i for *one* such i into three non-empty subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of odd cardinalities, and we partition all the other S_i as in [BKLR, (12.4)]. However, this partitioning of the sets S_i into three subsets does not satisfy [BKLR, (12.3)] for *other* i such that Δ_i is of type \mathbf{D}_{l_i} with odd l_i . This inaccuracy can be easily corrected: we should partition S_i for *each* i such that Δ_i is of type \mathbf{D}_{l_i} with odd l_i into three non-empty subsets of odd cardinalities.

3 MORE NON-QUASI-PERMUTATION LATTICES

In this section we construct another family of non-quasi-permutation lattices.

3.1. For $i = 1, \dots, r$ let $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ denote the root lattice and the weight lattice of \mathbf{SL}_{n_i} , respectively, and let $W_i = \mathfrak{S}_{n_i}$ denote the corresponding Weyl group (the symmetric group on n_i letters) acting on P_i and Q_i . Set $F_i = P_i/Q_i$, then W_i acts trivially on F_i . Set

$$Q = \bigoplus_{i=1}^r Q_i, \quad P = \bigoplus_{i=1}^r P_i, \quad W = \prod_{i=1}^r W_i,$$

then $Q \subset P$ and the Weyl group W acts on Q and P . Set

$$F = P/Q = \bigoplus_{i=1}^r F_i,$$

then W acts trivially on F .

We regard $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ as the lattices described in Bourbaki [Bou, Planche I]. Then we have an isomorphism $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$. Note that for each $1 \leq i \leq r$, the set $\{\alpha_{\varkappa,i} \mid 1 \leq \varkappa \leq n_i - 1\}$ is a \mathbb{Z} -basis of Q_i .

Set $c = \gcd(n_1, \dots, n_r)$; we assume that $c > 1$. Let $d > 1$ be a divisor of c . For each $i = 1, \dots, r$, let $\nu_i \in \mathbb{Z}$ be such that $1 \leq \nu_i < d$, $\gcd(\nu_i, d) = 1$, and assume that $\nu_1 = 1$. We write $\boldsymbol{\nu} = (\nu_i)_{i=1}^r \in \mathbb{Z}^r$. Let $\overline{\boldsymbol{\nu}}$ denote the image of $\boldsymbol{\nu}$ in $(\mathbb{Z}/d\mathbb{Z})^r$. Let $S_{\boldsymbol{\nu}} \subset (\mathbb{Z}/d\mathbb{Z})^r \subset \bigoplus_{i=1}^r \mathbb{Z}/n_i\mathbb{Z} = F$ denote the cyclic subgroup of order d generated by $\overline{\boldsymbol{\nu}}$. Let $L_{\boldsymbol{\nu}}$ denote the preimage of $S_{\boldsymbol{\nu}} \subset F$ in P under the canonical epimorphism $P \twoheadrightarrow F$, then $Q \subset L_{\boldsymbol{\nu}} \subset P$.

PROPOSITION 3.2. *Let W and the W -lattice $L_{\boldsymbol{\nu}}$ be as in Subsection 3.1. In the case $d = 2^s$ we assume that $\sum n_i > 4$. Then $L_{\boldsymbol{\nu}}$ is not quasi-permutation.*

This proposition follows from Lemmas 3.3 and 3.8 below.

LEMMA 3.3. *Let $p|d$ be a prime. Then for any subgroup $\Gamma \subset W$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m , the Γ -lattices $L_{\boldsymbol{\nu}}$ and $L_{\mathbf{1}} := L_{(1,\dots,1)}$ are equivalent for any $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$ as above (in particular, we assume that $\nu_1 = 1$).*

Note that this lemma is trivial when $d = 2$.

3.4. We compute the lattice $L_{\boldsymbol{\nu}}$ explicitly. First let $r = 1$. We have $Q = Q_1$, $P = P_1$. Then P_1 is generated by Q_1 and an element $\omega \in P_1$ whose image in P_1/Q_1 is of order n_1 . We may take

$$\omega = \frac{1}{n_1}[(n_1 - 1)\alpha_1 + (n_1 - 2)\alpha_2 + \dots + 2\alpha_{n_1-2} + \alpha_{n_1-1}],$$

where $\alpha_1, \dots, \alpha_{n_1-1}$ are the simple roots, see [Bou, Planche I]. There exists exactly one intermediate lattice L between Q_1 and P_1 such that $[L : Q_1] = d$, and it is generated by Q_1 and the element

$$w = \frac{n_1}{d}\omega = \frac{1}{d}[(n_1 - 1)\alpha_1 + (n_1 - 2)\alpha_2 + \dots + 2\alpha_{n_1-2} + \alpha_{n_1-1}].$$

Now for any natural r , the lattice $L_{\boldsymbol{\nu}}$ is generated by Q and the element

$$w_{\boldsymbol{\nu}} = \frac{1}{d} \sum_{i=1}^r \nu_i [(n_i - 1)\alpha_{1,i} + (n_i - 2)\alpha_{2,i} + \dots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

In particular, $L_{\mathbf{1}}$ is generated by Q and

$$w_{\mathbf{1}} = \frac{1}{d} \sum_{i=1}^r [(n_i - 1)\alpha_{1,i} + (n_i - 2)\alpha_{2,i} + \dots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

3.5. *Proof of Lemma 3.3.* Recall that $L_\nu = \langle Q, w_\nu \rangle$ with

$$Q = \langle \alpha_{\varkappa, i} \rangle, \quad \text{where } i = 1, \dots, r, \varkappa = 1, \dots, n_i - 1.$$

Set $Q_\nu = \langle \nu_i \alpha_{\varkappa, i} \rangle$. Denote by \mathfrak{T}_ν the endomorphism of Q that acts on Q_i by multiplication by ν_i . We have $Q_1 = Q$, $Q_\nu = \mathfrak{T}_\nu Q_1$, $w_\nu = \mathfrak{T}_\nu w_1$. Consider

$$\mathfrak{T}_\nu L_1 = \langle Q_\nu, w_\nu \rangle.$$

Clearly the W -lattices L_1 and $\mathfrak{T}_\nu L_1$ are isomorphic. We have an embedding of W -lattices $Q \hookrightarrow L_\nu$, in particular, an embedding $Q \hookrightarrow L_1$, which induces an embedding $\mathfrak{T}_\nu Q \hookrightarrow \mathfrak{T}_\nu L_1$. Set $M_\nu = L_\nu / \mathfrak{T}_\nu L_1$, then we obtain a homomorphism of W -modules $Q / \mathfrak{T}_\nu Q \rightarrow M_\nu$, which is an isomorphism by Lemma 3.6 below.

Now let $p|d$ be a prime. Let $\Gamma \subset W$ be a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m . As in [LPR, Proof of Proposition 2.10], we use Roiter's version [Ro, Proposition 2] of Schanuel's lemma. We have exact sequences of Γ -modules

$$\begin{aligned} 0 \rightarrow \mathfrak{T}_\nu L_1 \rightarrow L_\nu \rightarrow M_\nu \rightarrow 0, \\ 0 \rightarrow Q \xrightarrow{\mathfrak{T}_\nu} Q \rightarrow M_\nu \rightarrow 0. \end{aligned}$$

Since all ν_i are prime to p , we have $|\Gamma| \cdot M_\nu = p^m M_\nu = M_\nu$, and by [Ro, Corollary of Proposition 3] the morphisms of $\mathbb{Z}[\Gamma]$ -modules $L_\nu \rightarrow M_\nu$ and $Q \rightarrow M_\nu$ are projective in the sense of [Ro, §1]. Now by [Ro, Proposition 2] there exists an isomorphism of Γ -lattices $L_\nu \oplus Q \simeq \mathfrak{T}_\nu L_1 \oplus Q$. Since Q is a quasi-permutation W -lattice, it is a quasi-permutation Γ -lattice, and by Lemma 3.7 below, $L_\nu \sim \mathfrak{T}_\nu L_1$ as Γ -lattices. Since $\mathfrak{T}_\nu L_1 \simeq L_1$, we conclude that $L_\nu \sim L_1$. \square

LEMMA 3.6. *With the above notation $L_\nu / \mathfrak{T}_\nu L_1 \simeq Q / \mathfrak{T}_\nu Q = \bigoplus_{i=2}^r Q_i / \nu_i Q_i$.*

Proof. We have $\mathfrak{T}_\nu L_1 = \langle S_\nu \rangle$, where $S_\nu = \{\nu_i \alpha_{\varkappa, i}\}_{i, \varkappa} \cup \{w_\nu\}$. Note that

$$dw_\nu = \sum_{i=1}^r \nu_i [(n_i - 1)\alpha_{1, i} + (n_i - 2)\alpha_{2, i} + \dots + 2\alpha_{n_i - 2, i} + \alpha_{n_i - 1, i}].$$

We see that dw_ν is a linear combination with integer coefficients of $\nu_i \alpha_{\varkappa, i}$ and that $\alpha_{n_1 - 1, 1}$ appears in this linear combination with coefficient 1 (because $\nu_1 = 1$). Set $B'_\nu = S_\nu \setminus \{\alpha_{n_1 - 1, 1}\}$, then $\langle B'_\nu \rangle \ni \alpha_{n_1 - 1, 1}$, hence $\langle B'_\nu \rangle = \langle S_\nu \rangle = \mathfrak{T}_\nu L_1$, thus B'_ν is a basis of $\mathfrak{T}_\nu L_1$. Similarly, the set $B_\nu := \{\alpha_{\varkappa, i}\}_{i, \varkappa} \cup \{w_\nu\} \setminus \{\alpha_{n_1 - 1, 1}\}$ is a basis of L_ν . Both bases B_ν and B'_ν contain $\alpha_{1, 1}, \dots, \alpha_{n_1 - 2, 1}$ and w_ν . For all $i = 2, \dots, r$ and all $\varkappa = 1, \dots, n_i - 1$, the basis B_ν contains $\alpha_{\varkappa, i}$, while B'_ν contains $\nu_i \alpha_{\varkappa, i}$. We see that the homomorphism of W -modules $Q / \mathfrak{T}_\nu Q = \bigoplus_{i=2}^r Q_i / \nu_i Q_i \rightarrow L_\nu / \mathfrak{T}_\nu L_1$ is an isomorphism. \square

LEMMA 3.7. *Let Γ be a finite group, A and A' be Γ -lattices. If $A \oplus B \sim A' \oplus B'$, where B and B' are quasi-permutation Γ -lattices, then $A \sim A'$.*

Proof. Since B and B' are quasi-permutation, by Lemma 1.3 they are equivalent to 0, and we have

$$A = A \oplus 0 \sim A \oplus B \sim A' \oplus B' \sim A' \oplus 0 = A'.$$

This completes the proof of Lemma 3.7 and hence of Lemma 3.3. \square

To complete the proof of Proposition 3.2 it suffices to prove the next lemma.

LEMMA 3.8. *Let $p|d$ be a prime. Then there exists a subgroup $\Gamma \subset W$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m such that the Γ -lattice $L_1 := L_{(1,\dots,1)}$ is not quasi-permutation.*

3.9. Denote by U_i the space \mathbb{R}^{n_i} with canonical basis $\varepsilon_{1,i}, \varepsilon_{2,i}, \dots, \varepsilon_{n_i,i}$. Denote by V_i the subspace of codimension 1 in U_i consisting of vectors with zero sum of the coordinates. The group $W_i = \mathfrak{S}_{n_i}$ (the symmetric group) permutes the basis vectors $\varepsilon_{1,i}, \varepsilon_{2,i}, \dots, \varepsilon_{n_i,i}$ and thus acts on U_i and V_i . Consider the homomorphism of vector spaces

$$\chi_i: U_i \rightarrow \mathbb{R}, \quad \sum_{\lambda=1}^{n_i} \beta_{\lambda,i} \varepsilon_{\lambda,i} \mapsto \sum_{\lambda=1}^{n_i} \beta_{\lambda,i}$$

taking a vector to the sum of its coordinates. Clearly this homomorphism is W_i -equivariant, where W_i acts trivially on \mathbb{R} . We have short exact sequences

$$0 \rightarrow V_i \rightarrow U_i \xrightarrow{\chi_i} \mathbb{R} \rightarrow 0.$$

Set $U = \bigoplus_{i=1}^r U_i$, $V = \bigoplus_{i=1}^r V_i$. The group $W = \prod_{i=1}^r W_i$ naturally acts on U and V , and we have an exact sequence of W -spaces

$$0 \rightarrow V \rightarrow U \xrightarrow{\chi} \mathbb{R}^r \rightarrow 0, \quad (3.1)$$

where $\chi = (\chi_i)_{i=1,\dots,r}$ and W acts trivially on \mathbb{R}^r .

Set $n = \sum_{i=1}^r n_i$. Consider the vector space $\overline{U} := \mathbb{R}^n$ with canonical basis $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_n$. Consider the natural isomorphism

$$\varphi: U = \bigoplus_i U_i \xrightarrow{\sim} \overline{U}$$

that takes $\varepsilon_{1,1}, \varepsilon_{2,1}, \dots, \varepsilon_{n_1,1}$ to $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_{n_1}$, takes $\varepsilon_{1,2}, \varepsilon_{2,2}, \dots, \varepsilon_{n_2,2}$ to $\overline{\varepsilon}_{n_1+1}, \overline{\varepsilon}_{n_1+2}, \dots, \overline{\varepsilon}_{n_1+n_2}$, and so on. Let \overline{V} denote the subspace of codimension 1 in \overline{U} consisting of vectors with zero sum of the coordinates. Sequence (3.1) induces an exact sequence of W -spaces

$$0 \rightarrow \varphi(V) \rightarrow \overline{V} \xrightarrow{\psi} \mathbb{R}^r \xrightarrow{\Sigma} \mathbb{R} \rightarrow 0. \quad (3.2)$$

$$y = b\bar{w} + \sum_{j=1}^{n-1} a_j \bar{\alpha}_j$$

where $b, a_j \in \mathbb{Z}$, because $y \in \Lambda_n(d)$. We see that in the basis $\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}$ of $\Lambda_n(d) \otimes_{\mathbb{Z}} \mathbb{R}$, the element y contains $\bar{\alpha}_{j_i}$ with coefficient

$$b \frac{n - j_i}{d} + a_{j_i}.$$

Since $y \in \varphi(Q \otimes_{\mathbb{Z}} \mathbb{R})$, this coefficient must be 0:

$$b \frac{n - j_i}{d} + a_{j_i} = 0.$$

Consider

$$\begin{aligned} y - b\mu &= y - b \left(\bar{w} - \sum_{i=1}^{r-1} \frac{n - j_i}{d} \bar{\alpha}_{j_i} \right) = y - b\bar{w} + \sum_{i=1}^{r-1} \frac{b(n - j_i)}{d} \bar{\alpha}_{j_i} \\ &= \sum_{j=1}^{n-1} a_j \bar{\alpha}_j + \sum_{i=1}^{r-1} \frac{b(n - j_i)}{d} \bar{\alpha}_{j_i} = \sum_{j \in J} a_j \bar{\alpha}_j, \end{aligned}$$

where $a_j \in \mathbb{Z}$. We see that $y \in \langle \bar{\alpha}_j \ (j \in J), \mu \rangle$ for any $y \in N$, hence $N \subset \langle \bar{\alpha}_j \ (j \in J), \mu \rangle$. Conversely, $\mu \in N$ and $\bar{\alpha}_j \in N$ for $j \in J$, hence $\langle \bar{\alpha}_j \ (j \in J), \mu \rangle \subset N$, thus

$$N = \langle \bar{\alpha}_j \ (j \in J), \mu \rangle. \quad (3.3)$$

Now

$$\varphi(w) = \frac{1}{d} \left[\sum_{j=1}^{n_1-1} (n_1 - j) \bar{\alpha}_j + \sum_{j=1}^{n_2-1} (n_2 - j) \bar{\alpha}_{n_1+j} + \dots + \sum_{j=1}^{n_r-1} (n_r - j) \bar{\alpha}_{j_{r-1}+j} \right]$$

while

$$\mu = \frac{1}{d} \left[\sum_{j=1}^{n_1-1} (n - j) \bar{\alpha}_j + \sum_{j=1}^{n_2-1} (n - n_1 - j) \bar{\alpha}_{n_1+j} + \dots + \sum_{j=1}^{n_r-1} (n_r - j) \bar{\alpha}_{j_{r-1}+j} \right].$$

Thus

$$\mu = \varphi(w) + \frac{n - n_1}{d} \sum_{j=1}^{n_1-1} \bar{\alpha}_j + \frac{n - n_1 - n_2}{d} \sum_{j=1}^{n_2-1} \bar{\alpha}_{n_1+j} + \dots + \frac{n_r}{d} \sum_{j=1}^{n_r-1} \bar{\alpha}_{j_{r-1}+j},$$

where the coefficients

$$\frac{n - n_1}{d}, \quad \frac{n - n_1 - n_2}{d}, \quad \dots, \quad \frac{n_r}{d}$$

are integral. We see that

$$\langle \bar{\alpha}_j \ (j \in J), \mu \rangle = \langle \bar{\alpha}_j \ (j \in J), \varphi(w) \rangle. \quad (3.4)$$

From (3.3) and (3.4) we obtain that

$$N = \langle \bar{\alpha}_j \ (j \in J), \mu \rangle = \langle \bar{\alpha}_j \ (j \in J), \varphi(w) \rangle = \varphi(L). \quad \square$$

3.12. Now let $p \mid \gcd(n_1, \dots, n_r)$. Recall that $W = \prod_{i=1}^r \mathfrak{S}_{n_i}$. Since $p \mid n_i$ for all i , we can naturally embed $(\mathfrak{S}_p)^{n_i/p}$ into \mathfrak{S}_{n_i} . We obtain a natural embedding

$$\Gamma := (\mathbb{Z}/p\mathbb{Z})^{n/p} \hookrightarrow (\mathfrak{S}_p)^{n/p} \hookrightarrow W.$$

In order to prove Lemma 3.8, it suffices to prove the next Lemma 3.13. Indeed, if n has an odd prime factor p , then by Lemma 3.13 L is not quasi-permutation. If $n = 2^s$, then we take $p = 2$. By the assumptions of Proposition 3.2, $n > 4 = 2^2$, and again by Lemma 3.13 L is not quasi-permutation. This proves Lemma 3.8.

LEMMA 3.13. *If either p odd or $n > p^2$, then L is not quasi-permutation as a Γ -lattice.*

Proof. By Lemma 3.11 it suffices to show that N is not quasi-permutation. Since $N = \Lambda_n(d) \cap \varphi(V)$, we have an embedding

$$\Lambda_n(d)/N \hookrightarrow \overline{V}/\varphi(V).$$

By (3.2) $\overline{V}/\varphi(V) \simeq \mathbb{R}^{r-1}$ and W acts on $\overline{V}/\varphi(V)$ trivially. Thus $\Lambda_n(d)/N \simeq \mathbb{Z}^{r-1}$ and W acts on \mathbb{Z}^{r-1} trivially. We have an exact sequence of W -lattices

$$0 \rightarrow N \rightarrow \Lambda_n(d) \rightarrow \mathbb{Z}^{r-1} \rightarrow 0,$$

with trivial action of W on \mathbb{Z}^{r-1} . We obtain that $N \sim \Lambda_n(d)$ as a W -lattice, and hence, as a Γ -lattice. Therefore, it suffices to show that $\Lambda_n(d) = Q_n(n/d)$ is not quasi-permutation as a Γ -lattice if either p is odd or $n > p^2$. This is done in [LPR] in the proofs of Propositions 7.4 and 7.8. This completes the proof of Lemma 3.13 and hence those of Lemma 3.8 and Proposition 3.2. \square

4 QUASI-PERMUTATION LATTICES – CASE \mathbf{A}_{n-1}

In this section we prove Theorem 0.2 in the special case when G is isogenous to a direct product of groups of type \mathbf{A}_{n-1} for $n \geq 3$.

We maintain the notation of Subsection 3.1. Let L be an intermediate lattice between Q and P , i.e., $Q \subset L \subset P$ ($L = Q$ or $L = P$ are possible). Let S denote the image of L in F , then L is the preimage of $S \subset F$ in P . Since W acts trivially on F , the subgroup $S \subset F$ is W -invariant, and therefore, the sublattice $L \subset P$ is W -invariant.

THEOREM 4.1. *With the notation of Subsection 3.1 assume that $n_i \geq 3$ for all $i = 1, 2, \dots, r$. Let L between Q and P be an intermediate lattice, and assume that $L \cap P_i = Q_i$ for all i such that $n_i = 3$ or $n_i = 4$. If L is a quasi-permutation W -lattice, then $L = Q$.*

Proof. We prove the theorem by induction on r . The case $r = 1$ follows from our assumptions if $n_1 = 3$ or $n_1 = 4$, and from Proposition 1.9 if $n_1 > 4$.

We assume that $r > 1$ and that the assertion is true for $r - 1$. We prove it for r .

For i between 1 and r we set

$$Q'_i = \bigoplus_{j \neq i} Q_j, \quad P'_i = \bigoplus_{j \neq i} P_j, \quad F'_i = \bigoplus_{j \neq i} F_j, \quad W'_i = \prod_{j \neq i} W_j,$$

then $Q'_i \subset Q$, $P'_i \subset P$, $F'_i \subset F$ and $W'_i \subset W$. If L is a quasi-permutation W -lattice, then by Lemma 1.7 $L \cap P'_i$ is a quasi-permutation W'_i -lattice, and by the induction hypothesis $L \cap P'_i = Q'_i$.

Now let $Q \subset L \subset P$, and assume that $L \cap P'_i = Q'_i$ for all $i = 1, \dots, r$. We shall show that if $L \neq Q$ then L is not a quasi-permutation W -lattice. This will prove Theorem 4.1.

Assume that $L \neq Q$. Set $S = L/Q \subset F$, then $S \neq 0$. We first show that $(L \cap P'_i)/Q'_i = S \cap F'_i$. Indeed, clearly $(L \cap P'_i)/Q'_i \subset L/Q \cap P'_i/Q'_i = S \cap F'_i$. Conversely, let $f \in S \cap F'_i$, then f can be represented by some $l \in L$ and by some $p \in P'_i$, and $q := l - p \in Q$. Since $L \supset Q$, we see that $p = l - q \in L \cap P'_i$, hence $f \in (L \cap P'_i)/Q'_i$, and therefore $S \cap F'_i \subset (L \cap P'_i)/Q'_i$. Thus $(L \cap P'_i)/Q'_i = S \cap F'_i$.

By assumption we have $L \cap P'_i = Q'_i$, and we obtain that $S \cap F'_i = 0$ for all $i = 1, \dots, r$. Let $S_{(i)}$ denote the image of S under the projection $F \rightarrow F_i$. We have a canonical epimorphism $p_i: S \rightarrow S_{(i)}$ with kernel $S \cap F'_i$. Since $S \cap F'_i = 0$, we see that $p_i: S \rightarrow S_{(i)}$ is an isomorphism. Set $q_i = p_i \circ p_1^{-1}: S_{(1)} \rightarrow S_{(i)}$, it is an isomorphism.

We regard $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ as the lattices described in [Bou, Planche I]. Then we have an isomorphism $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$. Since $S_{(i)}$ is a subgroup of the cyclic group $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$ and $S \cong S_{(i)}$, we see that S is a cyclic group, and we see also that $|S|$ divides n_i for all i , hence $d := |S|$ divides $c := \gcd(n_1, \dots, n_r)$.

We describe all subgroups S of order d in $\bigoplus_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$ such that $S \cap (\bigoplus_{j \neq i} \mathbb{Z}/n_j\mathbb{Z}) = 0$ for all i . The element $a_i := n_i/d + n_i\mathbb{Z}$ is a generator of $S_{(i)} \subset F_i = \mathbb{Z}/n_i\mathbb{Z}$. Set $b_i = q_i(a_1)$. Since b_i is a generator of $S_{(i)}$, we have $b_i = \bar{\nu}_i a_i$ for some $\bar{\nu}_i \in (\mathbb{Z}/d\mathbb{Z})^\times$. Let $\nu_i \in \mathbb{Z}$ be a representative of $\bar{\nu}_i$ such that $1 \leq \nu_i < d$, then $\gcd(\nu_i, d) = 1$. Moreover, since $q_1 = \text{id}$, we have $b_1 = a_1$, hence $\bar{\nu}_1 = 1$ and $\nu_1 = 1$. We obtain an element $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$. With the notation of Subsection 3.1 we have $S = S_{\boldsymbol{\nu}}$ and $L = L_{\boldsymbol{\nu}}$.

By Proposition 3.2 $L_{\boldsymbol{\nu}}$ is not a quasi-permutation W -lattice. Thus L is not quasi-permutation, which completes the proof of Theorem 4.1. \square

5 PROOF OF THEOREM 0.2

LEMMA 5.1 (well-known). *Let P_1 and P_2 be abelian groups. Set $P = P_1 \oplus P_2 = P_1 \times P_2$, and let $\pi_1: P \rightarrow P_1$ denote the canonical projection. Let $L \subset P$ be a*

subgroup. If $\pi_1(L) = L \cap P_1$, then

$$L = (L \cap P_1) \oplus (L \cap P_2).$$

Proof. Let $x \in L$. Set $x_1 = \pi_1(x) \in \pi_1(L)$. Since $\pi_1(L) = L \cap P_1$, we have $x_1 \in L \cap P_1$. Set $x_2 = x - x_1$, then $x_2 \in L \cap P_2$. We have $x = x_1 + x_2$. This completes the proof of Lemma 5.1. \square

5.2. Let I be a finite set. For any $i \in I$ let D_i be a connected Dynkin diagram. Let $D = \bigsqcup_i D_i$ (disjoint union). Let Q_i and P_i be the root and weight lattices of D_i , respectively, and W_i be the Weyl group of D_i . Set

$$Q = \bigoplus_{i \in I} Q_i, \quad P = \bigoplus_{i \in I} P_i, \quad W = \prod_{i \in I} W_i.$$

5.3. We construct certain quasi-permutation lattices L such that $Q \subset L \subset P$.

Let $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$ be a set of non-ordered pairs in I such that D_{i_l} and D_{j_l} for all $l = 1, \dots, s$ are of type $\mathbf{B}_1 = \mathbf{A}_1$ and all the indices $i_1, j_1, \dots, i_s, j_s$ are distinct. Fix such an l . We write $\{i, j\}$ for $\{i_l, j_l\}$ and we set $D_{i,j} := D_i \sqcup D_j$, $Q_{i,j} := Q_i \oplus Q_j$, $P_{i,j} := P_i \oplus P_j$. We regard $D_{i,j}$ as a Dynkin diagram of type \mathbf{D}_2 , and we denote by $M_{i,j}$ the intermediate lattice between $Q_{i,j}$ and $P_{i,j}$ isomorphic to $\mathcal{X}(\mathbf{SO}_4)$, the character lattice of the group \mathbf{SO}_4 ; see Section 1, after Lemma 1.7. Let f_i be a generator of the lattice Q_i of rank 1, and let f_j be a generator of Q_j , then $P_i = \langle \frac{1}{2}f_i \rangle$ and $P_j = \langle \frac{1}{2}f_j \rangle$. Set $e_1^{(l)} = \frac{1}{2}(f_i + f_j)$, $e_2^{(l)} = \frac{1}{2}(f_i - f_j)$, then $\{e_1^{(l)}, e_2^{(l)}\}$ is a basis of $M_{i,j}$, and

$$M_{i,j} = \left\langle Q_{i,j}, e_1^{(l)} \right\rangle, \quad P_{i,j} = \left\langle M_{i,j}, \frac{1}{2}(e_1^{(l)} + e_2^{(l)}) \right\rangle. \quad (5.1)$$

We have $M_{i,j} \cap P_i = Q_i$, $M_{i,j} \cap P_j = Q_j$, and $[M_{i,j} : Q_{i,j}] = 2$. Concerning the Weyl group, we have

$$W(D_{i,j}) = W(D_i) \times W(D_j) = W(\mathbf{D}_2) = \mathfrak{S}_2 \times \{\pm 1\},$$

where the symmetric group \mathfrak{S}_2 permutes the basis vectors $e_1^{(l)}$ and $e_2^{(l)}$ of $M_{i,j}$, while the group $\{\pm 1\}$ acts on $M_{i,j}$ by multiplication by scalars. We say that $M_{i,j}$ is an *indecomposable quasi-permutation lattice* (it corresponds to the semisimple Cayley group \mathbf{SO}_4 which does not decompose into a direct product of its normal subgroups).

Set $I' = I \setminus \bigcup_{l=1}^s \{i_l, j_l\}$. For $i \in I'$ let M_i be any quasi-permutation intermediate lattice between Q_i and P_i (such an intermediate lattice exists if and only if D_i is of one of the types \mathbf{A}_n , \mathbf{B}_n , \mathbf{C}_n , \mathbf{D}_n , \mathbf{G}_2 , see Proposition 1.9). We say that M_i is a *simple quasi-permutation lattice* (it corresponds to a stably Cayley simple group). We set

$$L = \bigoplus_{l=1}^s M_{i_l, j_l} \oplus \bigoplus_{i \in I'} M_i. \quad (5.2)$$

We say that a lattice L as in (5.2) is a *direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices*. Clearly L is a quasi-permutation W -lattice.

THEOREM 5.4. *Let D, Q, P, W be as in Subsection 5.2. Let L be an intermediate lattice between Q and P , i.e., $Q \subset L \subset P$ (where $L = Q$ and $L = P$ are possible). If L is a quasi-permutation W -lattice, then L is as in (5.2). Namely, then L is a direct sum of indecomposable quasi-permutation lattices $M_{i,j}$ for some set of pairs $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$ and some family of simple quasi-permutation intermediate lattices M_i between Q_i and P_i for $i \in I'$.*

Remark 5.5. The set of pairs $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$ in Theorem 5.4 is uniquely determined by L . Namely, a pair $\{i, j\}$ belongs to this set if and only if the Dynkin diagrams D_i and D_j are of type $\mathbf{B}_1 = \mathbf{A}_1$ and

$$L \cap P_i = Q_i, \quad L \cap P_j = Q_j, \quad \text{while } L \cap (P_i \oplus P_j) \neq Q_i \oplus Q_j.$$

Proof of Theorem 5.4. We prove the theorem by induction on $m = |I|$, where I is as in Subsection 5.2. The case $m = 1$ is trivial.

We assume that $m \geq 2$ and that the theorem is proved for all $m' < m$. We prove it for m . First we consider three special cases.

Split case. Assume that for some subset $A \subset I$, $A \neq I$, $A \neq \emptyset$, we have $\pi_A(L) = L \cap P_A$, where $P_A = \bigoplus_{i \in A} P_i$ and $\pi_A: P \rightarrow P_A$ is the canonical projection. Then by Lemma 5.1 we have $L = (L \cap P_A) \oplus (L \cap P_{A'})$, where $A' = I \setminus A$. By Lemma 1.7 $L \cap P_A$ is a quasi-permutation W_A -lattice, where $W_A = \prod_{i \in A} W_i$. By the induction hypothesis the lattice $L \cap P_A$ is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices. Similarly, $L \cap P_{A'}$ is such a direct sum. We conclude that $L = (L \cap P_A) \oplus (L \cap P_{A'})$ is such a direct sum, and we are done.

\mathbf{A}_{n-1} -case. Assume that all D_i are of type \mathbf{A}_{n_i-1} , where $n_i \geq 3$ (so \mathbf{A}_1 is not permitted). We assume also that when $n_i = 3$ and when $n_i = 4$ (that is, for \mathbf{A}_2 and for $\mathbf{A}_3 = \mathbf{D}_3$) we have $L \cap P_i = Q_i$ (for $n_i > 4$ this is automatic because $L \cap P_i$ is a quasi-permutation W_i -lattice, see Proposition 1.9). In this case by Theorem 4.1 we have $L = Q = \bigoplus Q_i$, hence L is a direct sum of simple quasi-permutation lattices, and we are done.

\mathbf{A}_1 -case. Assume that all D_i are of type \mathbf{A}_1 . Then by [BKLR, Theorem 18.1] the lattice L is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices, and we are done.

Now we shall show that these three special cases exhaust all the quasi-permutation lattices. In other words, we shall show that if $Q \subset L \subset P$ and L is not as in one of these three cases, then L is not quasi-permutation. This will complete the proof of the theorem.

For the sake of contradiction, let us assume that $Q \subset L \subset P$, that L is not in one of the three special cases above, and that L is a quasi-permutation W -lattice.

We shall show in three steps that L is as in Proposition 2.2. By Proposition 2.2, L is not quasi-permutation, which contradicts our assumptions. This contradiction will prove the theorem.

Step 1. For $i \in I$ consider the intersection $L \cap P_i$, it is a quasi-permutation W_i -lattice (by Lemma 1.7), hence D_i is of one of the types \mathbf{A}_{n-1} , \mathbf{B}_n , \mathbf{C}_n , \mathbf{D}_n , \mathbf{G}_2 (by Proposition 1.9). Note that $\pi_i(L) \neq L \cap P_i$ (otherwise we are in the split case).

Now assume that for some $i \in I$, the Dynkin diagram D_i is of type \mathbf{G}_2 or \mathbf{C}_n for some $n \geq 3$, or D_i is of type \mathbf{A}_2 and $L \cap P_i \neq Q_i$. Then $L \cap P_i$ is a quasi-permutation W_i -lattice (by Lemma 1.7), hence $L \cap P_i = P_i$ (by Proposition 1.9). Since $P_i \supset \pi_i(L) \supset L \cap P_i$, we obtain that $\pi_i(L) = L \cap P_i$, which is impossible. Thus no D_i can be of type \mathbf{G}_2 or \mathbf{C}_n , $n \geq 3$, and if D_i is of type \mathbf{A}_2 for some i , then $L \cap P_i = Q_i$.

Thus all D_i are of types \mathbf{A}_{n-1} , \mathbf{B}_n or \mathbf{D}_n , and if D_i is of type \mathbf{A}_2 for some $i \in I$, then $L \cap P_i = Q_i$. Since L is not as in the \mathbf{A}_{n-1} -case, we may assume that one of the D_i , say D_1 , is of type \mathbf{B}_n for some $n \geq 1$ ($\mathbf{B}_1 = \mathbf{A}_1$ is permitted), or of type \mathbf{D}_n for some $n \geq 4$, or of type \mathbf{D}_3 with $L \cap P_1 \neq Q_1$. Indeed, otherwise all D_i are of type \mathbf{A}_{n_i-1} for $n_i \geq 3$, and in the cases \mathbf{A}_2 ($n_i = 3$) and \mathbf{A}_3 ($n_i = 4$) we have $L \cap P_i = Q_i$, i.e., we are in the \mathbf{A}_{n-1} -case, which contradicts our assumptions.

Step 2. In this step, using the Dynkin diagram D_1 of type \mathbf{B}_n or \mathbf{D}_n from the previous step, we construct a quasi-permutation sublattice $L' \subset L$ of index 2 such that L' is as in (5.2). First we consider the cases \mathbf{B}_n and \mathbf{D}_n separately.

Assume that D_1 is of type \mathbf{B}_n for some $n \geq 1$ ($\mathbf{B}_1 = \mathbf{A}_1$ is permitted). We have $[P_1 : Q_1] = 2$. Since $P_1 \supset \pi_1(L) \supsetneq L \cap P_1 \supset Q_1$, we see that $\pi_1(L) = P_1$ and $L \cap P_1 = Q_1$. Set $M_1 = Q_1$. We have $\pi_1(L) = P_1$, $L \cap P_1 = M_1$, and $[P_1 : M_1] = 2$.

Now assume that D_1 is of type \mathbf{D}_n for some $n \geq 4$, or of type \mathbf{D}_3 with $L \cap P_1 \neq Q_1$. Set $M_1 = L \cap P_1$, then M_1 is a quasi-permutation W_1 -lattice by Lemma 1.7, and it follows from Proposition 1.9 that $(W_1, M_1) \simeq \mathcal{X}(\mathbf{SO}_{2n})$, where $\mathcal{X}(\mathbf{SO}_{2n})$ denotes the character lattice of \mathbf{SO}_{2n} ; see Section 1, after Lemma 1.7. It follows that $[M_1 : Q_1] = 2$ and $[P_1 : M_1] = 2$. Since $P_1 \supset \pi_1(L) \supsetneq L \cap P_1 = M_1$, we see that $\pi_1(L) = P_1$. Again we have $\pi_1(L) = P_1$, $L \cap P_1 = M_1$, and $[P_1 : M_1] = 2$.

Now we consider together the cases when D_1 is of type \mathbf{B}_n for some $n \geq 1$ and when D_1 is of type \mathbf{D}_n for some $n \geq 3$, where in the case \mathbf{D}_3 we have $L \cap P_1 \neq Q_1$. Set

$$L' := \ker[L \xrightarrow{\pi_1} P_1 \rightarrow P_1/M_1].$$

Since $\pi_1(L) = P_1$, and $[P_1 : M_1] = 2$, we have $[L : L'] = 2$. Clearly we have $\pi_1(L') = M_1$. Set

$$L_1^\dagger := \ker[\pi_1 : L \rightarrow P_1] = L \cap P_1',$$

where $P'_1 = \bigoplus_{i \neq 1} P_i$. Since L is a quasi-permutation W -lattice, by Lemma 1.7 the lattice L_1^\dagger is a quasi-permutation W'_1 -lattice, where $W'_1 = \prod_{i \neq 1} W_i$. By the induction hypothesis, L_1^\dagger is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices as in (5.2). Since $M_1 = L \cap P_1$, we have $M_1 \subset L' \cap P_1$, and $L' \cap P_1 \subset L \cap P_1 = M_1$, hence $L' \cap P_1 = M_1 = \pi_1(L')$, and by Lemma 5.1 we have $L' = M_1 \oplus L_1^\dagger$. Since M_1 is a simple quasi-permutation lattice, we conclude that L' is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices as in (5.2), and $[L : L'] = 2$.

Step 3. In this step we show that L is as in Proposition 2.2. We write

$$L' = \bigoplus_{l=1}^s (L' \cap P_{i_l, j_l}) \oplus \bigoplus_{i \in I'} (L' \cap P_i),$$

where $P_{i_l, j_l} = P_{i_l} \oplus P_{j_l}$, the Dynkin diagrams D_{i_l} and D_{j_l} are of type $\mathbf{A}_1 = \mathbf{B}_1$, and $L' \cap P_{i_l, j_l} = M_{i_l, j_l}$ as in (5.1). For any $i \in I'$, we have $[\pi_i(L) : \pi_i(L')] \leq 2$, because $[L : L'] = 2$. Furthermore, for $i \in I'$ we have

$$\pi_i(L') = L' \cap P_i \subset L \cap P_i \subsetneq \pi_i(L),$$

hence $[\pi_i(L) : (L \cap P_i)] = 2$ and $L' \cap P_i = L \cap P_i$. Similarly, for any $l = 1, \dots, s$, if we write $i = i_l$, $j = j_l$, then we have

$$M_{i,j} = L' \cap P_{i,j} \subset L \cap P_{i,j} \subsetneq \pi_{i,j}(L) \subset P_{i,j}, \quad [P_{i,j} : M_{i,j}] = 2,$$

whence $\pi_{i,j}(L) = P_{i,j}$, $L \cap P_{i,j} = M_{i,j}$, and therefore $[\pi_{i,j}(L) : (L \cap P_{i,j})] = [P_{i,j} : M_{i,j}] = 2$ and $L' \cap P_{i,j} = M_{i,j} = L \cap P_{i,j}$.

We view the Dynkin diagram $D_{i_l} \sqcup D_{j_l}$ of type $\mathbf{A}_1 \sqcup \mathbf{A}_1$ corresponding to the pair $\{i_l, j_l\}$ ($l = 1, \dots, s$) as a Dynkin diagram of type \mathbf{D}_2 . Thus we view L' as a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices corresponding to Dynkin diagrams of type \mathbf{B}_n , \mathbf{D}_n and \mathbf{A}_n .

We wish to show that L is as in Proposition 2.2. We change our notation in order to make it closer to that of Proposition 2.2.

As in Subsection 2.1, we now write D_i for Dynkin diagrams of types \mathbf{B}_{l_i} and \mathbf{D}_{l_i} only, appearing in L' , where $\mathbf{B}_1 = \mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{C}_2$, $\mathbf{D}_2 = \mathbf{A}_1 \sqcup \mathbf{A}_1$ and $\mathbf{D}_3 = \mathbf{A}_3$ are permitted, but for \mathbf{D}_{l_i} with $l_i = 2, 3$ we require that

$$L \cap P_i = M_i := \mathcal{X}(\mathbf{SO}_{2l_i}).$$

We write $L'_i := L \cap P_i = L' \cap P_i$. We have $[\pi_i(L) : L'_i] = 2$, hence $[P_i : L'_i] \geq 2$. If D_i is of type \mathbf{B}_{l_i} , then $[P_i : L'_i] = 2$. If D_i is of type \mathbf{D}_{l_i} , then $L'_i = L \cap P_i \neq Q_i$, for \mathbf{D}_2 and \mathbf{D}_3 by our assumption and for \mathbf{D}_{l_i} with $l_i \geq 4$ because $L \cap P_i$ is a quasi-permutation W_i -lattice (see Proposition 1.9); again we have $[P_i : L'_i] = 2$.

We see that for all i we have $[P_i : L'_i] = 2$, $\pi_i(L) = P_i$, and the lattice $L'_i = M_i$ is as in Subsection 2.1. We realize the root system $R(D_i)$ of type \mathbf{B}_{l_i} or \mathbf{D}_{l_i} in the standard way (cf. [Bou, Planches II, IV]) in the space $V_i := \mathbb{R}^{l_i}$ with basis $(e_s)_{s \in S_i}$, then L'_i is the lattice generated by the basis vectors $(e_s)_{s \in S_i}$ of V_i , and we have $P_i = \langle L'_i, \frac{1}{2}x_i \rangle$, where

$$x_i = \sum_{s \in S_i} e_s \in L'_i.$$

In particular, when D_i is of type \mathbf{D}_2 we have $x_i = e_1^{(l)} + e_2^{(l)}$ with the notation of formula (5.1).

As in Subsection 2.1, we write Δ_ι for Dynkin diagrams of type $\mathbf{A}_{n'_\iota-1}$ appearing in L' , where $n'_\iota \geq 3$ and for $\mathbf{A}_3 = \mathbf{D}_3$ we require that $L \cap P_\iota = Q_\iota$. We write $L'_\iota := L \cap P_\iota = L' \cap P_\iota$. Then $L'_\iota = Q_\iota$ for all ι , for \mathbf{A}_2 by Step 1, for \mathbf{A}_3 by our assumption, and for other $\mathbf{A}_{n'_\iota-1}$ because L'_ι is a quasi-permutation W_ι -lattice; see Proposition 1.9. We have $\pi_\iota(L) \supsetneq L \cap P_\iota = L'_\iota$ and $[\pi_\iota(L) : L'_\iota] = [\pi_\iota(L) : \pi_\iota(L')] \leq 2$ (because $[L : L'] = 2$). It follows that $[\pi_\iota(L) : L'_\iota] = 2$, i.e., $[\pi_\iota(L) : Q_\iota] = 2$. We know that P_ι/Q_ι is a cyclic group of order n'_ι . Since it has a subgroup $\pi_\iota(L)/Q_\iota$ of order 2, we conclude that n'_ι is even, $n'_\iota = 2n_\iota$ (where $2n_\iota \geq 4$), and $\pi_\iota(L)/Q_\iota$ is the unique subgroup of order 2 of the cyclic group P_ι/Q_ι of order $2n_\iota$. As in Subsection 2.1, we realize the root system Δ_ι of type $\mathbf{A}_{2n_\iota-1}$ in the standard way (cf. [Bou, Planche I]) in the subspace V_ι of vectors with zero sum of the coordinates in the space \mathbb{R}^{2n_ι} with basis $\varepsilon_{\iota,1}, \dots, \varepsilon_{\iota,2n_\iota}$. We set

$$\xi_\iota = \varepsilon_{\iota,1} - \varepsilon_{\iota,2} + \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \dots + \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota},$$

then $\xi_\iota \in L'_\iota$ and $\frac{1}{2}\xi_\iota \in \pi_\iota(L) \setminus L'_\iota$ (cf. [Bou, Planche I, formula (VI)]), hence $\pi_\iota(L) = \langle L'_\iota, \frac{1}{2}\xi_\iota \rangle$.

Now we set

$$v = \frac{1}{2} \sum_{i \in I} x_i + \frac{1}{2} \sum_{\iota=1}^{\mu} \xi_\iota.$$

We claim that

$$L = \langle L', v \rangle.$$

Proof of the claim. Let $w \in L \setminus L'$, then $L = \langle L', w \rangle$, because $[L : L'] = 2$. Set $z_i = \frac{1}{2}x_i - \pi_i(w)$, then $z_i \in L'_i \subset L'$, because $\frac{1}{2}x_i, \pi_i(w) \in \pi_i(L) \setminus L'_i$. Similarly, we set $\zeta_\iota = \frac{1}{2}\xi_\iota - \pi_\iota(w)$, then $\zeta_\iota \in L'_\iota \subset L'$. We see that

$$v = w + \sum_i z_i + \sum_\iota \zeta_\iota,$$

where $\sum_i z_i + \sum_\iota \zeta_\iota \in L'$, and the claim follows. \square

It follows from the claim that L is as in Proposition 2.2 (we use the assumption that we are not in the \mathbf{A}_1 -case). Now by Proposition 2.2 L is not quasi-invertible, hence not quasi-permutation, which contradicts our assumptions. This contradiction proves Theorem 5.4. \square

Proof of Theorem 0.2. Theorem 0.2 follows immediately from Theorem 5.4 by virtue of Proposition 1.8. \square

6 PROOF OF THEOREM 0.3

In this section we deduce Theorem 0.3 from Theorem 0.2.

Let G be a stably Cayley semisimple k -group. Then $\overline{G} := G \times_k \bar{k}$ is stably Cayley over an algebraic closure \bar{k} of k . By Theorem 0.2, $G_{\bar{k}} = \prod_{j \in J} G_{j, \bar{k}}$ for some finite index set J , where each $G_{j, \bar{k}}$ is either a stably Cayley simple group or is isomorphic to $\mathbf{SO}_{4, \bar{k}}$. (Recall that $\mathbf{SO}_{4, \bar{k}}$ is stably Cayley and semisimple, but is not simple.) Here we write $G_{j, \bar{k}}$ for the factors in order to emphasize that they are defined over \bar{k} . By Remark 5.5 the collection of direct factors $G_{j, \bar{k}}$ is determined uniquely by \overline{G} . The Galois group $\text{Gal}(\bar{k}/k)$ acts on $G_{\bar{k}}$, hence on J . Let Ω denote the set of orbits of $\text{Gal}(\bar{k}/k)$ in J . For $\omega \in \Omega$ set $G_{\bar{k}}^{\omega} = \prod_{j \in \omega} G_{j, \bar{k}}$, then $\overline{G} = \prod_{\omega \in \Omega} G_{\bar{k}}^{\omega}$. Each $G_{\bar{k}}^{\omega}$ is $\text{Gal}(\bar{k}/k)$ -invariant, hence it defines a k -form G_k^{ω} of $G_{\bar{k}}^{\omega}$. We have $G = \prod_{\omega \in \Omega} G_k^{\omega}$.

For each $\omega \in \Omega$ choose $j = j_{\omega} \in \omega$. Let l_j/k denote the Galois extension in \bar{k} corresponding to the stabilizer of j in $\text{Gal}(\bar{k}/k)$. The subgroup $G_{j, \bar{k}}$ is $\text{Gal}(\bar{k}/l_j)$ -invariant, hence it comes from an l_j -form G_{j, l_j} . By the definition of Weil's restriction of scalars (see e.g. [Vo2, Subsection 3.12]) $G_{\bar{k}}^{\omega} \cong R_{l_j/k} G_{j, l_j}$, hence $G \cong \prod_{\omega \in \Omega} R_{l_j/k} G_{j, l_j}$. Each G_{j, l_j} is either absolutely simple or an l_j -form of \mathbf{SO}_4 .

We complete the proof using an argument from [BKLR, Proof of Lemma 11.1]. We show that G_{j, l_j} is a direct factor of $G_{l_j} := G \times_k l_j$. It is clear from the definition that $G_{j, \bar{k}}$ is a direct factor of $G_{\bar{k}}$ with complement $G'_{\bar{k}} = \prod_{i \in J \setminus \{j\}} G_{i, \bar{k}}$. Then $G'_{\bar{k}}$ is $\text{Gal}(\bar{k}/l_j)$ -invariant, hence it comes from some l_j -group G'_{l_j} . We have $G_{l_j} = G_{j, l_j} \times_{l_j} G'_{l_j}$, hence G_{j, l_j} is a direct factor of G_{l_j} .

Recall that G_{j, l_j} is either a form of \mathbf{SO}_4 or absolutely simple. If it is a form of \mathbf{SO}_4 , then clearly it is stably Cayley over l_j . It remains to show that if G_{j, l_j} is absolutely simple, then G_{j, l_j} is stably Cayley over l_j . The group $G_{\bar{k}}$ is stably Cayley over \bar{k} . Since $G_{j, \bar{k}}$ is a direct factor of the stably Cayley \bar{k} -group $G_{\bar{k}}$ over the algebraically closed field \bar{k} , by [LPR, Lemma 4.7] $G_{j, \bar{k}}$ is stably Cayley over \bar{k} . Comparing [LPR, Theorem 1.28] and [BKLR, Theorem 1.4], we see that G_{j, l_j} is either stably Cayley over l_j (in which case we are done) or an outer form of \mathbf{PGL}_{2n} for some $n \geq 2$. Thus assume by way of contradiction that G_{j, l_j} is an outer form of \mathbf{PGL}_{2n} for some $n \geq 2$. Then by [BKLR, Example 10.7] the character lattice of G_{j, l_j} is not quasi-invertible, and by [BKLR, Proposition

10.8] the group G_{j,l_j} cannot be a direct factor of a stably Cayley l_j -group. This contradicts the fact that G_{j,l_j} is a direct factor of the stably Cayley l_j -group G_{l_j} . We conclude that G_{j,l_j} cannot be an outer form of \mathbf{PGL}_{2n} for any $n \geq 2$. Thus G_{j,l_j} is stably Cayley over l_j , as desired. \square

A APPENDIX: SOME QUASI-PERMUTATION CHARACTER LATTICES

The positive assertion of Proposition 1.9 above is well known. It is contained in [CK, Theorem 0.1] and in [BKLR, Theorem 1.4]. However, [BKLR] refers to [CK, Theorem 0.1], and [CK] refers to a series of results on rationality (rather than only stable rationality) of corresponding generic tori. In this appendix for the reader's convenience we provide a proof of the following positive result in terms of lattices only.

PROPOSITION A.1. *Let G be any form of one of the following groups*

$$\mathbf{SL}_3, \mathbf{PGL}_n \text{ (} n \text{ odd)}, \mathbf{SO}_n \text{ (} n \geq 3), \mathbf{Sp}_{2n}, \mathbf{G}_2$$

or an inner form of \mathbf{PGL}_n (} n \text{ even)}. Then the character lattice of G is quasi-permutation.

Proof. \mathbf{SO}_{2n+1} . Let L be the character lattice of \mathbf{SO}_{2n+1} (including \mathbf{SO}_3). Then the Dynkin diagram is $D = \mathbf{B}_n$. The Weyl group is $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Then $L = \mathbb{Z}^n$ with the standard basis e_1, \dots, e_n . The group \mathfrak{S}_n naturally permutes e_1, \dots, e_n , while $(\mathbb{Z}/2\mathbb{Z})^n$ acts by sign changes. Since W permutes the basis up to \pm sign, the W -lattice L is quasi-permutation, see [Lo, § 2.8].

\mathbf{SO}_{2n} , any form, inner or outer. Let L be the character lattice of \mathbf{SO}_{2n} (including \mathbf{SO}_4). Then the Dynkin diagram is $D = \mathbf{D}_n$, with root system $R = R(D)$. We consider the pair (A, L) where $A = \text{Aut}(R, L)$, then (A, L) is isomorphic to the character lattice of \mathbf{SO}_{2n+1} , hence is quasi-permutation.

\mathbf{Sp}_{2n} . The character lattice of \mathbf{Sp}_{2n} is isomorphic to the character lattice of \mathbf{SO}_{2n+1} , hence is quasi-permutation.

\mathbf{PGL}_n , inner form. The character lattice of \mathbf{PGL}_n is the root lattice $L = Q$ of \mathbf{A}_{n-1} . It is a quasi-permutation \mathfrak{S}_n -lattice, cf. [Lo, Example 2.8.1].

\mathbf{PGL}_n , outer form, n odd. Let P be the weight lattice of \mathbf{A}_{n-1} , where $n \geq 3$ is odd. Then P is generated by elements e_1, \dots, e_n subject to the relation

$$e_1 + \dots + e_n = 0.$$

The automorphism group $A = \text{Aut}(\mathbf{A}_{n-1})$ is the product of \mathfrak{S}_n and \mathfrak{S}_2 . The group A acts on P as follows: \mathfrak{S}_n permutes e_1, \dots, e_n , and the nontrivial element of \mathfrak{S}_2 takes each e_i to $-e_i$.

We denote by M the A -lattice of rank $2n+1$ with basis $s_1, \dots, s_n, t_1, \dots, t_n, u$. The group \mathfrak{S}_n permutes s_i and permutes t_i ($i = 1, \dots, n$), and the nontrivial

element of \mathfrak{S}_2 permutes s_i and t_i for each i . The group A acts trivially on u . Clearly M is a permutation lattice.

We define an A -epimorphism $\pi: M \rightarrow P$ as follows:

$$\pi: \quad s_i \mapsto e_i, \quad t_i \mapsto -e_i, \quad u \mapsto 0.$$

Set $M' = \ker \pi$, it is an A -lattice of rank $n+2$. We show that it is a permutation lattice. We write down a set of $n+3$ generators of M' :

$$\rho_i = s_i + t_i, \quad \sigma = s_1 + \cdots + s_n, \quad \tau = t_1 + \cdots + t_n, \quad u.$$

There is a relation

$$\rho_1 + \cdots + \rho_n = \sigma + \tau.$$

We define a new set of $n+2$ generators:

$$\tilde{\rho}_i = \rho_i + u, \quad \tilde{\sigma} = \sigma + \frac{n-1}{2}u, \quad \tilde{\tau} = \tau + \frac{n-1}{2}u,$$

where $\frac{n-1}{2}$ is integral because n is odd. We have

$$\tilde{\rho}_1 + \cdots + \tilde{\rho}_n - \tilde{\sigma} - \tilde{\tau} = u,$$

hence this new set indeed generates M' , hence it is a basis. The group \mathfrak{S}_n permutes $\tilde{\rho}_1, \dots, \tilde{\rho}_n$, while \mathfrak{S}_2 permutes $\tilde{\sigma}$ and $\tilde{\tau}$. Thus A permutes our basis, and therefore M' is a permutation lattice. We have constructed a left resolution of P :

$$0 \rightarrow M' \rightarrow M \rightarrow P \rightarrow 0,$$

(with permutation lattices M and M'), which by duality gives a right resolution of the root lattice $Q \cong P^\vee$ of \mathbf{A}_{n-1} :

$$0 \rightarrow Q \rightarrow M^\vee \rightarrow (M')^\vee \rightarrow 0$$

with permutation lattices M^\vee and $(M')^\vee$. Thus the character lattice Q of \mathbf{PGL}_n is a quasi-permutation A -lattice for odd n .

The assertion that the character lattice of G is quasi-permutation in the remaining cases \mathbf{SL}_3 and \mathbf{G}_2 follows from the next Lemma A.2.

LEMMA A.2 ([BKLR, Lemma 2.5]). *Let Γ be a finite group and L be any Γ -lattice of rank $r = 1$ or 2 . Then L is quasi-permutation.*

This lemma, which is a version of [Vo2, § 4.9, Examples 6 and 7], was stated in [BKLR] without proof. For the sake of completeness we supply a short proof here.

We may assume that Γ is a maximal finite subgroup of $\mathbf{GL}_r(\mathbb{Z})$. If $r = 1$, then $\mathbf{GL}_1(\mathbb{Z}) = \{\pm 1\}$, and the lemma reduces to the case of the character lattice of \mathbf{SO}_3 treated above.

Now let $r = 2$. Up to conjugation there are two maximal finite subgroups of $\mathbf{GL}_2(\mathbb{Z})$, they are isomorphic to the dihedral groups D_8 (of order 8) and to D_{12} (of order 12), resp., see e.g. [Lo, § 1.10.1, Table 1.2]. The group D_8 is the group of symmetries of a square, and in this case it suffices to show that the character lattice of \mathbf{SO}_5 is quasi-permutation, which we have done above. The group D_{12} is the group of symmetries of a regular hexagon, and in this case it suffices to show that the character lattice of \mathbf{PGL}_3 (outer form) is quasi-permutation, which we have done above as well. This completes the proofs of Lemma A.2 and Proposition A.1. \square

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